A Galois correspondence for compact group actions on C*-algebras

Costel Peligrad

Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, OH 45221-0025. Email: peligrc@ucmail.uc.edu

Abstract. In this paper, we prove a Galois correspondence for compact group actions on C*-algebras in the presence of a commuting minimal action. Namely, we show that there is a one to one correspondence between the C*-subalgebras that are globally invariant under the compact action and the commuting minimal action, that in addition contain the fixed point algebra of the compact action and the closed, normal subgroups of the compact group.

Key words: C*-algebra, automorphism group, minimal action.
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1 Introduction

Let (M, G, δ) be a W*-dynamical system with M a von Neumann algebra, G a compact group and $\delta : g \in G \to \delta_g \in Aut(M)$ a homomorphism of G into the group $Aut(M)$ of all automorphisms of M such that the mapping $g \to \delta_g(m)$ is simple-weakly continuous for every $m \in M$. Denote by $Aut_\delta(M)$ the subgroup of $Aut(M)$ consisting of all automorphisms commuting with all $\delta_g, g \in G$. In [7], (see also [[15], Theorem 3]) it is proven that if $Aut_\delta(M)$ contains an ergodic subgroup $\mathcal{S}$, then there is a one to one correspondence between the set of normal, closed subgroups of G and the set of all G and $\mathcal{S}$ globally invariant von Neumann subalgebras N with $M^G \subset N \subset M$. This correspondence is given by: $N \longleftrightarrow G^N$ where $G^N = \{g \in G | \delta_g(n) = n, n \in N\}$. The main technical tool in Kishimoto's approach is the method of Hilbert spaces inside a von Neumann algebra, as developed in [14]. Later, in [6], the case of irreducible actions was considered. They proved that if $M^G \subset M$ is an irreducible pair of factors, i.e. $(M^G)' \cap M = \mathcal{C}I$, then there is a one to one correspondence between the set of intermediate subfactors $M^G \subset N \subset M$ and the closed subgroups of G given by $N \longleftrightarrow G^N$, where, as above, $G^N = \{g \in G | \delta_g(n) = n, n \in N\}$. This correspondence is called Galois correspondence. In [6], an action of a compact group with the property $(M^G)' \cap M = \mathcal{C}I$ is called minimal. Notice that in this case, $\mathcal{S} = \left\{Ad(u) | u \in M^G, \text{unitary}\right\}$ is ergodic on M and that N is obviously

$\mathcal{S}$-invariant but is not required to be $G-$invariant. This result was extended to the case of compact quantum group actions on von Neumann factors by Tomatsu [16]. Both papers, [6] and [16] make extensive use of the method of Hilbert spaces inside a von Neumann algebra and other methods specific for von Neumann algebras. In this paper, we will prove a result that extends Kishimoto's result to the case of compact actions on C*-algebras commuting with minimal actions, as defined below. To the best of our knowledge, this is the first result of this kind for C*-dynamical systems. The notion of minimal action that will be used in this paper is different from the one used in [6]. Our methods are specific to C*-dynamical systems and give, in paticular, a new proof of Kishimoto's result. We also give an example that shows that the result is not true if the commutant of the compact action satisfies a weaker ergodicity condition, that, in the case of von Neumann algebras is equivalent with the usual one.

2 Notations and preliminary results

2.1 Ergodic actions on C*-algebras

If M is a von Neumann algebra, a subgroup $\mathcal{S} \subset Aut(M)$ is called ergodic if $M^{\mathcal{S}} = \mathcal{C}I$, where $\mathcal{C}$ is the set of complex numbers and $M^{\mathcal{S}}$ denotes the fixed point algebra, $M^{\mathcal{S}} = \{m \in M | s(m) = m, s \in \mathcal{S}\}$. In the case of C*-algebras there are several distinct notions of ergodicity that are all equivalent for von Neumann algebras. These notions are distinct even for abelian C*-algebras, the case of topological dynamics. Let A be a C*-algebra and $\mathcal{S} \subset Aut(A)$ a subgroup of the automorphism group of A. Denote by $\mathcal{H}^{\mathcal{S}}(A)$ the set of all non zero hereditary C*-subalgebras of A that are globally $\mathcal{S}$-invariant. We recall the following definitions from [9]:

1) $\mathcal{S}$ is called weakly ergodic if $A^{\mathcal{S}}$ is trivial.

2) $\mathcal{S}$ is called topologically transitive if for every C_1, $C_2 \in \mathcal{H}^{\mathcal{S}}(A)$, their product $C_1C_2 = \left\{\sum_{finite} c_1^i c_2^i | c_1^i \in C_1, c_2^i \in C_2\right\}$ is not zero. In the particular case of topological dynamics this condition is equivalent to the usual topological transitivity of the flow.

In [2] it is noticed that our condition 2) is equivalent to the following:

2') If $x, y \in A$ are not zero, then there is an $s \in \mathcal{S}$ such that $xs(y) \neq 0$.

3) $\mathcal{S}$ is called minimal if $\mathcal{H}^{\mathcal{S}}(A) = \{A\}$.

We caution that in [6] and [16] the notion of minimality is used for compact actions (M, G, δ) such that $(M^G)' \cap M = CI$. In this paper, a group of automorphisms is called minimal if it satisfies condition 3) above.

Obviously 3)$\Longrightarrow$ 2)$\Longrightarrow$ 1). Also, since in the case of von Neumann algebras, M, the $\mathcal{S}$-invariant, hereditary W*-subalgebras, are of the form pMp where p is a projection in $M^{\mathcal{S}}$ it follows that all the above conditions are equivalent. Another situation when all of the above conditions are equivalent for a C*-dynamical system is when $\mathcal{S}$ is compact [[9], Corollary 2.7.].

In [9] there are also discussed several criteria for checking topological transitivity. In [2], a seemingly stronger notion than topological transitivity is introduced, namely the notion of strong topological transitivity:

$\mathcal{S}$ is said to be strongly topologically transitive if for each finite sequence $\{(x_i, y_i)|i = 1, 2, ..., n\}$ of pairs of elements $x_i, y_i \in A$ for which $\sum x_i \otimes y_i \neq 0$ in the algebraic tensor product $A \otimes A$, there exists an $s \in \mathcal{S}$ such that $\sum x_i s(y_i) \neq 0$ in A.

Further, in [[3], Theorem 5.2.] it is shown that in the case of von Neumann algebras strong topological transitivity is equivalent with topological transitivity and hence with the rest of the above conditions.

In what follows we will need the following results from [1]:

Proposition 1 *Let $(A, \mathcal{S})$ be a C*-dynamical system and $B \subset A$ an $\mathcal{S}$-invariant C*-subalgebra. Then $\mathcal{H}^{\mathcal{S}}(B) = \left\{C \cap B | C \in \mathcal{H}^{\mathcal{S}}(A)\right\}$.*

Proof. This is [[1], Proposition 2.3.]. ∎

If A is a C*-algebra, we denote by A_{sa} the set of selfadjoint elements of A and by $(A_{sa})^m$ the set of elements in the bidual A^{**} of A that can be obtained as strong limits of bounded, monotone increasing nets from A_{sa} [see also [10], 3.11]. Then we can state [1]:

Proposition 2 *Let $(A, \mathcal{S})$ be a C*-dynamical system. Then the following conditions are equivalent:*
i) $(A, \mathcal{S})$ is minimal
ii) If $a \in (A_{sa})^m$ is such that $s^{**}(a) = a$ for every $s \in \mathcal{S}$, then $a \in \mathcal{C}I$. Here, s^{**} denotes the double dual of the automorphism $s \in \mathcal{S}$.

Proof. This is [[1], Proposition 4.1.]. ∎

2.2 Compact group actions on C*-algebras

Let (A, G, δ) be a C*-dynamical system with G compact. Denote by $\widehat{G}$ the set of all equivalence classes of irreducible, unitary representations of G. For each $\pi \in \widehat{G}$, fix a unitary representation, u^π in the class π and a basis in the Hilbert space H_π of u^π. If $\pi \in \widehat{G}$, denote by $\chi_\pi(g) = d_\pi \sum_{i=1}^{d_\pi} \overline{u_{ii}^\pi(g)}$ the character of the class π, where d_π is the dimension of H_π. For $\pi \in \widehat{G}$ we consider the following mappings from A into itself:

$$P^{\pi,\delta}(a) = \int_G \chi_\pi(g) \delta_g(a) dg$$

$$P_{ij}^{\pi,\delta}(a) = \int_G \overline{u_{ji}^\pi}(g) \delta_g(a) dg$$

We define the spectral subspaces of the action δ:

$$A_1^\delta(\pi) = \left\{a \in A | P^{\pi,\delta}(a) = a\right\}, \pi \in \widehat{G}$$

In the particular case when $\pi = \pi_0$ is the trivial one dimensional representation of G, $A_1^\delta(\pi_0) = A^G$ the C*-subalgebra of fixed elements of the action δ.

As in [8] and [11], the matricial spectral subspaces are defined as follows:

$$A_2^\delta(u^\pi) = \{X = [x_{ij}] \in A \otimes B(H_\pi) | [\delta_g(x_{ij})] = X(1 \otimes u^\pi(g))\}$$

Notice that $A_2^\delta(u^\pi)$ depends on the representation u^π but for two equivalent representations, $A_2^\delta(u^\pi)$ are spatially isomorphic. Obviously, $A_2^\delta(u^\pi)A_2^\delta(u^\pi)^*$ is a two sided ideal of $A^G \otimes B(H_\pi)$ and $A_2^\delta(u^\pi)^* A_2^\delta(u^\pi)$ is a two sided ideal of $(A \otimes B(H_\pi))^{\delta \otimes ad(u^\pi)}$. The proof of the following remark is straightforward:

Remark 3 *Let (A, G, δ) be a C*-dynamical system with G compact and $s \in Aut(A)$ be such that $s\delta_g = \delta_g s$ for every $g \in G$. Then $s(A_1^\delta(\pi)) \subset A_1^\delta(\pi)$ and $(s \otimes \iota)(A_2^\delta(u^\pi)) \subset A_2^\delta(u^\pi)$ for every $\pi \in \widehat{G}$. Here ι stands for the identity automorphism of $B(H_\pi)$.*

We will use the following results from [11]:

Lemma 4 *i) $\sum_{\pi \in \widehat{G}} A_1^\delta(\pi)$ is dense in A*
ii) $A_2^\delta(u^\pi) = \left\{[P_{ij}^{\pi,\delta}(a)] | a \in A\right\}$

Proof. i) This is [[11], Lemma 2.3.]
ii) This is [[11], Lemma 2.2.] ■
We also need the following known result:

Lemma 5 *Let (C, G, δ) be a C*-dynamical system with G compact. Then every approximate unit of the fixed point algebra C^G is an approximate unit of C.*

Proof. See for instance [[5], Lemma 2.7.] for the more general case of compact quantum group actions. ■

Finally, we recall that a C*-dynamical system (A, G, δ) with G compact is called saturated if the closed, two sided ideal of the crossed product, $A \rtimes_\delta G$, generated by χ_{π_0} equals the crossed product. In this definition we used the known fact that every character, χ_π, of G is an element of the multiplier algebra $M(A \rtimes_\delta G)$ of the crossed product [[8], [11]]. If the system is saturated then, the crossed product is strongly Morita equivalent, in the sense of Rieffel, with the fixed point algebra, A^G [[13], page 236]. Then, we have [[11], Corollary 3.5.]:

Lemma 6 *Let (C, G, δ) be a C*-dynamical system with G compact. Then the following conditions are equivalent:*
i) The system is saturated
ii) The two sided ideal $C_2^\delta(u^\pi)^* C_2^\delta(u^\pi)$ is dense in $(C \otimes B(H_\pi))^{\delta \otimes ad(u^\pi)}$ for every $\pi \in \widehat{G}$, where $(C \otimes B(H_\pi))^{\delta \otimes ad(u^\pi)}$ is the fixed point algebra of $C \otimes B(H_\pi)$ for the action $\delta \otimes ad(u^\pi)$ of G.

3 Galois correspondence

In this section we will prove our main results and give examples and counterexamples.

Let (A, G, δ) be a C*-dynamical system with G compact. Let $B \subset A$ be G-invariant C*-subalgebra such that $A^G \subset B$.

Lemma 7 *If (B, G, δ) is saturated and if $\delta|_B$ is faithful, then $B = A$.*

Proof. By Lemma 4 ii), we have to prove that for every $\pi \in \widehat{G}$, $A_2^\delta(u^\pi) \subset B_2^\delta(u^\pi)$. Let $\pi \in \widehat{G}$ be arbitrary. Since (B, G, δ) is saturated, by Lemma 6 we have $\overline{B_2^\delta(u^\pi)^* B_2^\delta(u^\pi)} = (B \otimes B(H_\pi))^{\delta \otimes ad(u^\pi)}$. Since $(A^G \otimes I_{B(H_\pi)}) \subset (B \otimes B(H_\pi))^{\delta \otimes ad(u^\pi)}$, it follows from Lemma 5 that $(B \otimes B(H_\pi))^{\delta \otimes ad(u^\pi)}$ contains an approximate unit of $A \otimes B(H_\pi)$. Since $B_2^\delta(u^\pi)^* B_2^\delta(u^\pi)$ is a dense two sided ideal of $(B \otimes B(H_\pi))^{\delta \otimes ad(u^\pi)}$, by [[4], Proposition 1.7.2.], this latter C*-algebra contains an approximate unit $\{E_\lambda\} \subset B_2^\delta(u^\pi)^* B_2^\delta(u^\pi)$, $E_\lambda = \sum_{i=1}^{n_\lambda} X_{i,\lambda}^* Y_{i,\lambda}$ where $X_{i,\lambda}, Y_{i,\lambda} \in B_2^\delta(u^\pi)$. Let now $X \in A_2^\delta(u^\pi)$. Then $X E_\lambda = \sum X X_{i,\lambda}^* Y_{i,\lambda} = \sum (X X_{i,\lambda}^*) Y_{i,\lambda} \in (A^G \otimes B(H_\pi)) B_2^\delta(u^\pi) = B_2^\delta(u^\pi)$. Since $\{E_\lambda\}$ is an approximate unit of $A \otimes B(H_\pi)$, it follows that $X = (norm) \lim (X E_\lambda) \in B_2^\delta(u^\pi)$. Therefore $B = A$. ■

Let B be a G-invariant C*-subalgebra of A such that $A^G \subset B$. Denote $G^B = \{g \in G | \delta_g(b) = b, b \in B\}$. Then we have:

Remark 8 *i) G^B is a closed, normal subgroup of G*
ii) The quotient action $\delta^\bullet$ of G/G^B on B is faithful.

Proof. Straightforward. ■

Corollary 9 *Let (A, G, δ) be a C*-dynamical system. If $A^G \subset B \subset A$ and B is a G-invariant C*-subalgebra such that $(B, G/G^B, \delta^\cdot)$ is saturated, where $\delta^\bullet$ is the quotient action, then $B = A^{G^B}$.*

Proof. By Remark 8 ii) the quotient action $\delta^\bullet$ of G/G^B on B is faithful and therefore, if we apply Lemma 7 to G/G^B instead of G, we get the desired result. ■

Remark 10 *As we have noticed in the proof of the previous Lemma, if B is a G-invariant C*-subalgebra of A and $\pi \in \widehat{G}$ is such that $B_1^\delta(\pi) \neq (0)$ and hence $B_2^\delta(u^\pi) \neq (0)$, it follows that the ideal $\overline{B_2^\delta(u^\pi)^* B_2^\delta(u^\pi)}$ contains an approximate unit $\{E_\lambda\}$ of the form $E_\lambda = \sum_{i=1}^{n_\lambda} X_{i,\lambda}^* Y_{i,\lambda}$ where $X_{i,\lambda}, Y_{i,\lambda} \in B_2^\delta(u^\pi)$.*

Lemma 11 *Let A be a C*-algebra and $B \subset A$ a C*-subalgebra If $\mathcal{S} \subset Aut(A)$ acts minimally on A and leaves B globally invariant, then $\mathcal{S}$ acts minimally on B.*

Proof. Indeed, by Proposition 1 every $\mathcal{S}$-invariant hereditary C*-subalgebra of B, C, is the intersection of B with an $\mathcal{S}$-invariant hereditary subalgebra of A.Since $\mathcal{S}$ acts minimally on A it follows that $C = B$. ■

In what follows, we will need the following result from [12]:

Proposition 12 *Let (A, G, δ) be a dynamical system with G compact. Assume that the action δ is faithful and that there is a subgroup $\mathcal{S}$ of $Aut_\delta(A)$ which acts minimally on A. Then $A_1^\delta(\pi) \neq (0)$ for every $\pi \in \widehat{G}$.*

Proof. This is [[12], Proposition 5.2.]. ∎

The next lemma provides a class of C*-dynamical systems $(A, G.\delta)$ that are saturated. We will denote by $Aut_\delta(A)$ the subgroup of the group of all automorphisms of A consisting of all automorphisms that commute with δ_g for all $g \in G$.

Lemma 13 *Let (A, G, δ) be a dynamical system with G compact. Assume that the action δ is faithful and that there is a subgroup $\mathcal{S}$ of $Aut_\delta(A)$ which acts minimally on A.Then the system is saturated.*

Proof. We will prove that $\overline{A_2^\delta(u^\pi)^* A_2^\delta(u^\pi)} = A \otimes B(H_\pi)^{\alpha \otimes adu^\pi}$ for every $\pi \in \widehat{G}$ and the result will follow from Lemma 6. Notice first that according to Proposition 12, $A_1^\delta(\pi) \neq (0)$ for every $\pi \in \widehat{G}$ and hence $A_2^\delta(u^\pi) \neq (0)$ for every $\pi \in \widehat{G}$. Let $\pi \in \widehat{G}$ be arbitrary. Applying Remark 10, let $E_\lambda = \sum_{i=1}^{n_\lambda} X_{i,\lambda}^* Y_{i,\lambda}$, where $X_{i,\lambda}, Y_{i,\lambda} \in A_2^\delta(u^\pi)$, be an increasing approximate unit of $\overline{A_2^\delta(u^\pi)^* A_2^\delta(u^\pi)}$. Then $E_\lambda \nearrow E$ in the strong operator topology, where E is the unit of the von Neumann algebra generated by $\overline{A_2^\delta(u^\pi)^* A_2^\delta(u^\pi)}$ in $A^{**} \otimes B(H_\pi)$, where A^{**} is the second dual of A. Let H be the Hilbert space of the universal representation of A so that $A^{**} \subset B(H)$. Since every $s \in \mathcal{S}$ commutes with every δ_g, $g \in G$, by Remark 3 it follows that $A_2^\delta(u^\pi)^* A_2^\delta(u^\pi)$ and its weak closure in $A^{**} \otimes B(H_\pi)$ is globally invariant under the automorphisms $\{s^{**} \otimes \iota | s \in \mathcal{S}\}$ where s^{**} is the double dual of s and ι is the identity automorphism of $B(H_\pi)$. This means, in particular, that $(s^{**} \otimes \iota)(E) = E$ for every $s \in \mathcal{S}$. If we write $E_\lambda = \left[E_{ij}^\lambda\right]$ $i, j = 1, ..., d_\pi$ as a matrix with entries in A and $E = [E_{ij}]$, $E_{ij} \in A^{**}$, then $s^{**}(E_{ij}) = E_{ij}$ for every $s \in \mathcal{S}$ and all i, j. Since E is a projection, it is in particular a positive operator which is a strong limit of elements of $A \otimes B(H_\pi)$ Therefore every diagonal entry, E_{ii}, of E, is a positive operator which is the strong limit of an increasing net of positive elements of A, so $E_{ii} \in A^m$. By Proposition 2 it follows that there are scalars μ_{ii} such that $E_{ii} = \mu_{ii} I$ where I is the unit of $B(H)$. Now, $H \otimes H_\pi \simeq \oplus_{i=1}^{i=d_\pi} H_i$ where $H_i = H$ for all $i = 1, ...d_\pi$ with d_π the dimension of H_π. Let $\zeta = \oplus \zeta_i \in H \otimes H_\pi$ with $\zeta_{i_0} = \zeta_{j_0} = \xi \in H$ and $\zeta_i = 0$ if $j_0 \neq i \neq i_0$. Then we have

$$
\begin{aligned}
(E_\lambda \zeta, \zeta) = (E_{i_0 i_0}^\lambda \xi, \xi) + (E_{i_0 j_0}^\lambda \xi, \xi) + (E_{i_0 j_0}^{\lambda *} \xi, \xi) + (E_{j_0 j_0}^\lambda \xi, \xi) = \\
= ((E_{i_0 j_0}^\lambda + E_{i_0 j_0}^{\lambda *} + E_{i_0 i_0}^\lambda + E_{j_0 j_0}^\lambda) \xi, \xi)
\end{aligned}
$$

Since $E_\lambda \nearrow E$, it follows that $(E_{i_0 j_0}^\lambda + E_{i_0 j_0}^{\lambda *} + E_{i_0 i_0}^\lambda + E_{j_0 j_0}^\lambda) \nearrow (E_{i_0 j_0} + E_{i_0 j_0}^* + \mu_{i_0 i_0} I + \mu_{j_0 j_0} I)$ in the weak operator topology and, since $\{E_\lambda\}$ is norm bounded, in the strong operator toplogy. Hence, $E_{i_0 j_0} + E_{i_0 j_0}^* + \mu_{i_0 i_0} I + \mu_{j_0 j_0} I \in A^m$. As we noticed before, $E_{i_0 j_0} + E_{i_0 j_0}^* + \mu_{i_0 i_0} I + \mu_{j_0 j_0} I$ is s^{**}-invariant for every $s \in \mathcal{S}$

and therefore, by Proposition 2 it is a scalar multiple of the identity. Hence $E_{i_0j_0} + E^*_{i_0j_0}$ is a scalar multiple of the identity.

Similarly, considering $\zeta = \oplus\zeta_i \in H \otimes H_\pi$ with $\zeta_{i_0} = \sqrt{-1}\xi$, $\zeta_{j_0} = -\xi$, $\xi \in H$ and $\zeta_i = 0$ if $j_0 \neq i \neq i_0$ we infer that $E_{i_0j_0} - E^*_{i_0j_0}$ is a scalar multiple of the identity. Hence there are scalars μ_{ij} such that $E_{ij} = \mu_{ij}I$, so all entries of E are scalar multiples of the identity. Since E is an element of the weak closure of $A \otimes B(H_\pi)^{\delta\otimes ad(u^\pi)}$ it follows that E intertwines u^π with itself and therefore, since u^π is irreducible, we have $E = I$ and we are done. ■

We can now prove our main result:

Theorem 14 *Let (A, G, δ) be a dynamical system with G compact. Assume that there is a subgroup $\mathcal{S}$ of $Aut_\delta(A)$ which acts minimally on A. If $A^G \subset B \subset A$ and B is a G and $\mathcal{S}$ globally invariant C*-subalgebra, then $B = A^{G^B}$. Conversely, if $G_0 \subset G$ is a closed, normal subgroup, then $B = A^{G_0}$ is a G and $\mathcal{S}$ invariant C*-subalgebra such that $A^G \subset B \subset A$.*

Proof. It is immediate to see that the quotient action $\delta^\bullet$ acts faithfully on B. By Lemma 11, $\mathcal{S}$ acts minimally on B. By Lemma 13, the system $(B, \delta^\bullet, G/G^B)$ is saturated. By Corollary 1, $B = A^{G^B}$ and we are done. The converse is easily checked. ■

Notice that for W*-dynamical systems the proof of the above Theorem 14 is simpler since the discussion about lower semicontinuous elements in the bidual A^{**} is not necessary.

A simple example of a C*-dynamical system (A, G, δ) with G compact satisfying the hypotheses of Theorem 14 is the following:

Example 15 *Let G be a compact group and $C(G)$ the C*-algebra of continuous functions on G. Denote by λ the action of G on $C(G)$ by left translations and by ρ the action by right translations. Let H be a Hilbert space and $\mathcal{K}(H)$ the algebra of compact operators on H. Let $A = C(G)\otimes\mathcal{K}(H)$ and $\delta_g = \lambda_g \otimes \iota, g \in G$. Then the subgroup $\mathcal{S} \subset Aut_\delta(A)$ generated by $\left\{\rho_g \otimes ad(u) | g \in G, u \in \widetilde{\mathcal{K}(H)}, unitary\right\}$ where ρ_g is the right translation by $g \in G$, acts minimally on A. Here $\widetilde{\mathcal{K}(H)}$ denotes the C*-algebra obtained from $\mathcal{K}(H)$ by adjoining a unit if H is infinite dimensional.*

The next result provides a class of examples of C*-dynamical systems (A, G, δ) that satisfy the hypotheses of Theorem 14.

Theorem 16 *Let (A, G, δ) be a C*-dynamical system with G compact abelian. Assume that the fixed point algebra A^G is simple. If B is a G-invariant C*-subalgebra such that $A^G \subset B \subset A$, then $B = A^{G^B}$.*

Proof. Denote by $\widetilde{A^G}$ the C*-algebra obtained from A^G by adjoining a unit. We will show that the subgroup $\mathcal{S} \subset Aut(A)$ generated by $\delta_G = \{\delta_g | g \in G\}$ and $\left\{ad(u) | u \in \widetilde{A^G}, unitary\right\}$ is minimal. Since G is abelian and $ad(u)$, $u \in \widetilde{A^G}$

commute with $\delta_g, g \in G$, we have that $\mathcal{S} \subset Aut_\delta(A)$. We prove next that $\mathcal{S}$ acts minimally on A. Let $C \in \mathcal{H}^{\mathcal{S}}(A)$. Then if $L = \overline{AC}$, we have that L is an $\mathcal{S}$-invariant, in particular G-invariant, closed, left ideal of A and $C = L \cap L^*$. We show that $L = A$ and hence $C = A$. Since L is $\mathcal{S}$-invariant, it follows that $L^G = \left\{\int_G \delta_g(l)dg | l \in L\right\} \subset L \cap A^G$ is a left ideal of A^G. Since L is $ad(u)$-invariant for every $u \in \widetilde{A^G}$, unitary, we have:

$$L^G u = uad(u^*)(L^G) \subset L^G, u \in \widetilde{A^G} \text{ unitary.}$$

Therefore L^G is a two sided ideal of A^G. Since A^G is simple, it follows that $L^G = A^G$ and thus by Lemma 5, L^G and so L contains an approximate unit of A. Hence $L = A$ and therefore $C = L \cap L^* = A$. Therefore $\mathcal{S} \subset Aut_\delta(A)$ is minimal and the conclusion follows from Theorem 14. ■

An example of C*-dynamical system satisfying the hypotheses of Theorem 16 can be constructed as follows:

Example 17 *Let (C, G, λ) be a C*-dynamical system with G compact abelian. Assume that λ is weakly ergodic, and therefore minimal, by [[9], Corollary 2.7.]. Let H be a Hilbert space. Let $A = C \otimes \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the algebra of compact operators on H and $\delta_g = \lambda_g \otimes \iota, g \in G$ where ι is the trivial automorphism of $\mathcal{K}(H)$. Then (A, G, δ) satisfies the hypotheses of Theorem 16.*

Proof. Straightforward. ■

The next example shows that the conclusion of Theorem 14 may fail if the minimality condition on $\mathcal{S}$ is replaced with a weaker ergodicity condition such as topological transitivity, or even with strong topological transitivity.

Example 18 *Let G be a compact abelian group and H an infinit dimensional Hilbert space. Denote by τ the action of G, by translations, on $C(G)$, the C*-algebra of continuous functions on G. Let $A = C(G) \otimes \widetilde{\mathcal{K}(H)}$, where $\widetilde{\mathcal{K}(H)}$ is the subalgebra of $B(H)$ generated by $\mathcal{K}(H)$ and the unit $I \in B(H)$. Let $\delta_g = \tau_g \otimes \iota, g \in G$, where ι is the identity automorphism of $\widetilde{\mathcal{K}(H)}$. Consider the system (A, G, δ). Clearly, $A^G = I_{C(G)} \otimes \widetilde{\mathcal{K}(H)}$. We will prove the following two facts:*
i) $Aut_\delta(A)$ contains a subgroup $\mathcal{S}$ which acts strongly topologically transitively on A
*ii) There is a G and $\mathcal{S}$-invariant C*subalgebra B such that $A^G \subset B \subset A$ and $B \neq A^{G^B}$.*

Proof. i) Let $\mathcal{S} = \left\{\tau_g \otimes ad(u) | g \in G, u \in \widetilde{\mathcal{K}(H)}, unitary\right\} \subset Aut(A)$. Obviously, every element $s \in \mathcal{S}$ commutes with all $\delta_g = \tau_g \otimes \iota, g \in G$. We prove next that $\mathcal{S}$ acts ergodically on the von Neumann algebra $L^\infty(G) \otimes B(H)$ and then applying [[9], Theorem 2.2. i)] (respectively, [[3], Corollary 5.3.]) it will follow that $\mathcal{S}$ acts topologically transitively (strongly topologically transitively) on A. Notice first that τ_g is implemented by the unitary operator $\lambda_g \in B(L^2(G))$ of

translation by g. Hence the fixed point algebra $(B(L^2(G))\otimes B(H))^{\mathcal{S}}$ is the commutant $(C^*(G)^{''}\otimes B(H))^{'} = C^*(G)^{''}\otimes \mathcal{C}I$ where $C^*(G)$ is the group C*-algebra of G. Since $C^*(G)^{''}\cap L^\infty(G) = \mathcal{C}I$, i) is proven.
ii) Let $B\subset A$ be the C*-subalgebra generated by $C(G)\otimes\mathcal{K}(H)$ and $I_{C(G)}\otimes I_{B(H)}$. Then, B is obviously G and $\mathcal{S}$ invariant and $A^G = I_{C(G)}\otimes\widetilde{\mathcal{K}(H)}\subset B$. Clearly, $G^B = \{g\in G|\delta_g(b) = b, b\in B\} = \{e\}$ where e is the identity element of G. If we show that $B\neq A$, ii) is proven. Let $f\in C(G)$ be a non constant function. Then there are $g_1, g_2\in G$ such that $f(g_1)\neq f(g_2)$. We claim that $f\otimes I\notin B$. Assume to the contrary that $f\in B$. then there is a function $\Phi: G\to\mathcal{K}(H)$ and a scalar μ such that $f(g)\otimes I = \Phi(g)+\mu I$. In particular $(f(g_1)-f(g_2))I = \Phi(g_1)-\Phi(g_2)\in\mathcal{K}(H)$, which is a contradiction since $f(g_1)-f(g_2)$ is a non zero scalar. ■

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